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ON CONVEX FUNCTIONS

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1. We denote by K the class of analytic functions $f(z) = z + a_2z^2 + \ldots$ univalent and convex in the open unit disk D. Many properties of $f(z) \in K$ can be derived from Marx-Strohhäcker's theorem which states that for $f(z) \in K$ the inequalities

$$\text{(1)} \qquad \operatorname{Re}\frac{f(z)}{z} > \frac{1}{2}, \qquad \operatorname{Re}\frac{zf'(z)}{f(z)} > \frac{1}{2}, \quad z \in D,$$

hold.

The author [1] previously remarked that if $f(z) \in K$, then $f(z)$ is starlike with respect to symmetrical points, in other words

$$\text{(2)} \qquad \operatorname{Re}\frac{zf'(z)}{f(z) - f(-z)} > 0, \quad z \in D,$$

holds. This inequality means that for every r in $0 < r < 1$ the point $f(-z)$, $z = re^{i\theta}$, lies in the left half plane bounded by the directional tangent at the point $f(z)$ of the image curve of the circle $|z| = r$ under the function $f(z)$.

The above fact that all functions of K are starlike with respect to symmetrical points is evident by a brief geometrical consideration. But we shall give in Section 2 an analytical proof for this property of functions of K.

Now in this note we shall show that the condition (2) yields easily Marx-Strohhacker's theorem.

We first note that the condition (2) implies

$$\text{(3)} \qquad \frac{1-|z|}{2(1+|z|)} \leqq \operatorname{Re}\frac{zf'(z)}{f(z) - f(-z)} \leqq \frac{1+|z|}{2(1-|z|)}, \quad z \in D.$$

Suppose that $f(z) \in K$. Then for a such that $|a| < 1$ the function

$$F(z) = \frac{f((z+a)/(1+\bar{a}z)) - f(a)}{f'(a)(1 - |a|^2)} = z + \dots$$

also belongs to K. Therefore F(z) satisfies the condition (2). Hence from (3) we have

$$\text{(4)} \qquad \frac{1 - |z|}{2(1 + |z|)} \leqq \operatorname{Re} \frac{zf'\left(\dfrac{z+a}{1+\bar{a}z}\right) \dfrac{1 - |a|^2}{(1 + z\bar{a})^2}}{f\left(\dfrac{z+a}{1+\bar{a}z}\right) - f\left(\dfrac{-z+a}{1-\bar{a}z}\right)} \leqq \frac{1 + |z|}{2(1 - |z|)}, \quad z \in D.$$

Let z = a in (4). Then we have

$$\text{(5)} \qquad \frac{1 + |a|^2}{(1 + |a|)^2} \leqq \operatorname{Re} \frac{\dfrac{2a}{1 + |a|^2} f'\left(\dfrac{2a}{1+|a|^2}\right)}{f\left(\dfrac{2a}{1+|a|^2}\right)} \leqq \frac{1 + |a|^2}{(1 - |a|)^2}, \quad |a| < 1$$

Setting here $t = 2a/(1 + |a|^2)$, we find

$$\text{(6)} \qquad \frac{1}{1 + |t|} \leqq \operatorname{Re} \frac{tf'(t)}{f(t)} \leqq \frac{1}{1 - |t|}, \quad |t| < 1,$$

where t may take an arbitrary value in D. Hence we get the second inequality of (1).

Next we use the inequality

$$\text{(7)} \qquad \operatorname{Re} \frac{f(z) - f(-z)}{zf'(z)} > 0, \quad z \in D,$$

which is equivalent to (2), and implies

$$\text{(8)} \qquad \frac{2(1 - |z|)}{1 + |z|} \leqq \operatorname{Re} \frac{f(z) - f(-z)}{zf'(z)} \leqq \frac{2(1 + |z|)}{1 - |z|}, \quad z \in D.$$

Applying the condition (7) to F(z), from (8) we have

$$(9)\qquad \frac{2(1-|z|)}{1+|z|} \leqq \mathrm{Re}\,\frac{f\left(\dfrac{z+a}{1+\bar{a}z}\right)-f\left(\dfrac{-z+a}{1-\bar{a}z}\right)}{zf'\left(\dfrac{z+a}{1+\bar{a}z}\right)\dfrac{1-|a|^2}{(1+\bar{a}z)^2}} \leqq \frac{2(1+|z|)}{1-|z|},\quad z\in D.$$

Let $z = -a$ in (9). Then we have

$$(10)\qquad \frac{1+|a|^2}{(1+|a|)^2} \leqq \mathrm{Re}\,\frac{f\left(\dfrac{2a}{1+|a|^2}\right)}{\dfrac{2a}{1+|a|^2}} \leqq \frac{1+|a|^2}{(1-|a|)^2},\quad |a|<1.$$

Setting here $t = 2a/(1+|a|^2)$, we find similarly

$$(11)\qquad \frac{1}{1+|t|} \leqq \mathrm{Re}\,\frac{f(t)}{t} \leqq \frac{1}{1-|t|},\quad t\in D.$$

Hence we get the first inequality of (1). Thus our purpose is accomplished.

2. In this section we shall prove (2) for functions $f(z) \in K$ by an analytical method. The author and Watanabe [2] proved that if $\mathrm{Re}\{f'(z)/\phi'(z)\} > 0$, $|z| < 1$, for $\phi(z) \in K$, then $\mathrm{Re}\{(f(z) - f(\zeta))/(\phi(z) - \phi(\zeta))\} > 0$, $|z| < 1$, $|\zeta| < 1$, holds. From this it follows that if $f(z) \in K$, then

$$(12)\qquad \mathrm{Re}\,\frac{zf'(z)-\zeta f'(\zeta)}{f(z)-f(\zeta)} > 0,\quad |z|<1,\quad |\zeta|<1$$

holds. Because $\mathrm{Re}\{(zf'(z))'/f'(z)\} > 0$, $|z| < 1$, for $f \in K$.

Now let $f(z) \in K$. Then for an arbitrary a such that $|a| < 1$ the function

$$F(z) = \frac{f\left(\dfrac{z-a}{1-\bar{a}z}\right) - f(-a)}{f'(-a)(1-|a|^2)} = z + \dots$$

is also a member of K. Therefore from (12) we have

$$\text{(13)}\qquad \operatorname{Re} \frac{zf'\left(\dfrac{z-a}{1-\bar{a}z}\right)\dfrac{1-|a|^2}{(1-\bar{a}z)^2} - \zeta f'\left(\dfrac{\zeta-a}{1-\bar{a}\zeta}\right)\dfrac{1-|a|^2}{(1-\bar{a}\zeta)^2}}{f\left(\dfrac{z-a}{1-\bar{a}z}\right) - f\left(\dfrac{\zeta-a}{1-\bar{a}\zeta}\right)} > 0,$$

for $|z| < 1$, $|\zeta| < 1$. Putting $\zeta = 0$ and $z = 2a/(1 + |a|^2)$ in this inequality, we have

$$\operatorname{Re} \frac{\dfrac{2a}{1+|a|^2} f'(a)}{f(a) - f(-a)} > 0,$$

because $(z - a)/(1 - \bar{a}z) = a$. Hence

$$\text{(14)}\qquad \operatorname{Re} \frac{af'(a)}{f(a) - f(-a)} > 0,$$

so that (2) holds.

REFERENCES

[1] K. Sakaguchi, On certain univalent mapping, J. Math. Soc. Japan, 11(1959), 72 - 75.

[2] K. Sakaguchi and S. Watanabe, On close-to-convex functions, J. Nara Gakugei Univ., 14(1966), 7 - 12.

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